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The Haldane–Shastry model and RTT relations

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Abstract. A new approach is presented to derive the Hamiltonian family of spin-$\frac{1}{2}$ Haldane–Shastry model based on the RTT relation and Yangian symmetry. We show that the first members of the Hamiltonian family are related to the quantum determinant of the transfer matrix T.

1. Introduction

Recently there has been a remarkable success in studying the long-ranged interaction models by means of various approaches [1–9]. Besides the Calogero–Sutherland (C–S) type of models [1, 2, 5–8] the Haldane–Shastry (H–S) model was regarded as the representative of the spin chain $(\mathrm{SU}(n))$ with long-range interaction [3–6]. The first members of the Hamiltonian family of the H–S model are given by [3–5].

$$H_2 = \sum_{i,j}{}' \left(\frac{z_i z_j}{z_{ij} z_{ji}} \right) (P_{ij} - 1) \tag{1.1}$$

$$H_3 = \sum_{i,j,k}{}' \left(\frac{z_i z_j z_k}{z_{ij} z_{jk} z_{ki}} \right) (P_{ijk} - 1) \tag{1.2}$$

and empirically,

$$H_4 = \sum_{ijkl}{}' \left(\frac{z_i z_j z_k z_l}{z_{ij} z_{jk} z_{kl} z_{li}} \right) (P_{ijkl} - 1) + H_4^{'} \tag{1.3}$$

$$H_4^{'} = -\frac{1}{3} H_2 - 2 \sum_{i,j}{}' \left(\frac{z_i z_j}{z_{ij} z_{ji}} \right)^2 (P_{ij} - 1) \tag{1.4}$$

where P_{ij} exchange the states on sites i and j,

$$z_{ij} = z_i - z_j \tag{1.5}$$

with z_j being prime complex number. Following [3–5] we use the notations:

$$P_{ijk} = P_{ij} P_{jk} + P_{jk} P_{ki} + P_{ki} P_{ij} \tag{1.6}$$

$$P_{ijkl} = P_{ij} P_{jk} P_{kl} + \text{all the cyclic terms for } i \to j \to k \to l \tag{1.7}$$

$$[H_m, H_n] = 0 \qquad (m, n \leqslant 4) \tag{1.8}$$

$$[H_m, Q_1^a] = [H_m, Q_2^a] = 0 \tag{1.9}$$

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where

$$Q_0^a = \sum_{i=1}^{N} I_i^a \tag{1.10}$$

$$Q_1^a = \tfrac{1}{2}\sum_{i,j}{}' w_{ij} f^{abc} I_i^b I_j^c \tag{1.11}$$

and w_{ij} satisfy

$$w_{ij} = -w_{ji} \tag{1.12}$$

$$w_{ij}w_{ik} + w_{jk}w_{ji} + w_{ki}w_{ij} = 1 \tag{1.13}$$

whose general solution reads

$$w_{ij} = \frac{z_i + z_j}{z_i - z_j} \tag{1.14}$$

with a special form for N being the sum of the distinct sites:

$$z_j = \omega^j \qquad \omega = \exp\left(\frac{\mathrm{i}2\pi}{N}\right). \tag{1.15}$$

The Q_0^a and Q_1^a form an infinite algebra (Yangian) associated with the fundamental representations of $\mathrm{SU}(n)$, whose generators at i-site are I_i^a. The H–S model and its properties related to Yangian were systematically discussed in [3–5]. However, how to derive the Hamiltonian set equations (1.1)–(1.3) based on the RTT relation is left unsolved [1–8]. As is well known, the integrability in the sense of Yang–Baxter is very important. The key point consists of the RTT relation:

$$\check{R}(u-v)(T(u)\bigotimes T(v)) = (T(v)\bigotimes T(u))\check{R}(u-v) \tag{1.16}$$

where $\check{R}(u)$ satisfies the Yang–Baxter relations and $T(u)$ is the transfer matrix. Following the quantum inverse scattering methods [10] the $\mathrm{tr}\, T(u)$ forms a conserved family including the Hamiltonian that is related to the Bethe ansatz in the diagonalization. However, based on some long-ranged interaction models [6, 9], we have known that the quantum determinant $\mathrm{Det}\, T(u)$ may be regarded as the conserved family. An important property of $\mathrm{Det}\, T(u)$ is

$$[\mathrm{Det}\, T(u), T(v)] = 0 \tag{1.17}$$

i.e. commute with the quantum operators $T_{ab}(v)$. Because equations (1.1)–(1.3) commute with the generators of Yangian [3–5]. It follows from the Drinfeld theorem [11, 12] that the Hamiltonian family equations (1.1)–(1.3) should commute with $T_{ab}(u)$. A natural idea is to think that equations (1.1)–(1.3) may be derived through the $\mathrm{Det}\, T(u)$. However, as pointed out in [5], the origin of the Hamiltonian constants of motion in the H–S model is conceptually rather different from that in the finite-N Bethe model. In a special representation, the $\mathrm{Det}\, T(u)$ becomes a polynomial in u with constant coefficients. To avoid the quantum determinant generated by the transfer matrix $T(u)$ becoming a trivial c-number, we should find a different approach to generate the Hamiltonian family equations (1.1)–(1.3).

In this paper we would like to set up the connection between the H–S model and RTT relation. Essentially, this is to bridge $\mathrm{tr}\, T(u)$ and $\mathrm{Det}\, T(u)$. To overcome the difficulty stated above we should extend the new realization of Yangian proposed by Drinfeld [11, 12] by introducing $T_0(u) = \mathrm{tr}\, T(u)$ in the RTT relation. The approach can be viewed as an extension of [6, 9, 13] where the relation between the quantum determinant and the transfer matrix is given to determine the higher invariants for the type of C–S models.

From the expansion of $T(u)$

$$T(u) = \sum_{n=0}^{\infty} u^{-n} T^{(n)} \tag{1.18}$$

where $T^{(0)} = I$, in accordance with the RTT relation, the generators of Yangian are related to $T^{(1)}$ and $T^{(2)}$, whereas the $\operatorname{tr} T^{(m)} (m \geqslant 2)$ are constrained by the RTT relations with $n \geqslant 3$ (see below). Therefore one may find sufficient $\operatorname{tr} T^{(m)}$ $(m \geqslant 2)$ satisfying the first-few ordered expansion of RTT. They form a conserved family and do not commute with Yangian.

It is not difficult to set up the relationship between the quantum determinant $\operatorname{Det} T(u)$ and $\operatorname{tr} T^{(m)}$ family, namely, for given $\operatorname{tr} T^{(m)}$ satisfying RTT relation we are able to generate the Hamiltonian equations (1.1)–(1.3).

For simplicity we restricted ourselves to discuss the gl(2) case whose transfer matrix is simply determined by the R-matrix

$$\check{R}(u) = uP + I \tag{1.19}$$

where P stands for the 4×4 matrix representation of the permutation, and $T(u)$ takes the form

$$T(u) = \begin{bmatrix} T_{11}(u) & T_{12}(u) \\ T_{21}(u) & T_{22}(u) \end{bmatrix} = \sum_{n=0}^{\infty} u^{-n} T^{(n)} = \sum_{n=0}^{\infty} u^{-n} \| T_{ab}^{(n)} \| \tag{1.20}$$

where $a, b = 1, 2$, and $T_{ab}^{(n)}$ are quantum operators. Substituting equations (1.18), (1.19) and (1.20) into equation (1.16) we have

$$[T_{bc}^{(n+1)}, T_{ad}^{(m)}] - [T_{bc}^{(n)}, T_{ad}^{(m+1)}] + T_{ac}^{(n)} T_{bd}^{(m)} - T_{ac}^{(m)} T_{bd}^{(n)} = 0 \qquad (m, n \geqslant 0). \tag{1.21}$$

Defining

$$\begin{aligned} &T_0^{(n)} = \operatorname{tr} T^{(n)} = T_{11}^{(n)} + T_{22}^{(n)} \qquad T_3^{(n)} = T_{11}^{(n)} - T_{22}^{(n)} \\ &T_+^{(n)} = T_{12}^{(n)} \qquad T_-^{(n)} = T_{21}^{(n)} \end{aligned} \tag{1.22}$$

the quantum determinant

$$\operatorname{Det} T(u) = T_{11}(u) T_{22}(u-1) - T_{12}(u) T_{21}(u-1) \tag{1.23}$$

$$= \sum_{n=0}^{\infty} u^{-n} C_n \tag{1.24}$$

can be expressed through $T_0^{(n)}$ and $T_\alpha^{(m)}$ $(\alpha = \pm, 3)$:

$$\begin{aligned} C_n = T_0^{(n)} &+ \tfrac{1}{2} \sum_{m+k=n \ m,k \neq 0} C_{m,k} T_0^{(m)} + \tfrac{1}{2} \sum_{m+k+s=n \ k,m \neq 0} C_{m,s} \\ &\times [\tfrac{1}{2}(T_0^{(k)} T_0^{(m)} - T_3^{(k)} T_3^{(m)}) - (T_+^{(k)} T_-^{(m)} + T_-^{(k)} T_+^{(m)})] \end{aligned} \tag{1.25}$$

$$C_{m,k} = \frac{(k+m+1)!}{(m-1)!k!}.$$

It is noted that C_n are, in general, not constants (the explicit examples were given in [6, 9]). Obviously,

$$[C_m, C_n] = 0 \qquad [C_m, T_{ab}^{(m)}] = 0 \tag{1.26}$$

$$[T_0^{(m)}, T_0^{(n)}] = [T_0^{(m)}, C_n] = 0 \tag{1.27}$$

$$[T_0^{(n)}, T_{ab}^{(m)}] \neq 0 \qquad (m \geqslant 2). \tag{1.28}$$

In the following we shall find the $T_\alpha^{(n)}(\alpha = \pm, 3, 0)(n = 2, 3, 4)$ satisfying the RTT relation, then substitute them into equation (1.25) to generate $H_n(n = 2, 3, 4)$ through C_n.

This paper is organized as follows. In section 2 we shall give the RTT relations including $T_0^{(n)} = \operatorname{tr} T^{(n)}$. In section 3 the quantum determinant $\operatorname{Det} T(q)$ is constructed that will lead to the Hamiltonian set of the H–S model (1.1)–(1.4).

2. The commutation relations for $T_\alpha^{(m)}$ ($\alpha = \pm, 3, 0$)

With the 4×4 $\check{R}$-matrix equation (1.19) and the notions equation (1.22) the RTT relation equation (1.21) leads to the following independent relations:

$$[T_\alpha^{(1)}, T_\alpha^{(1)}] = [T_\alpha^{(1)}, T_\alpha^{(2)}] = [T_\alpha^{(2)}, T_\alpha^{(2)}] = 0 \qquad \alpha \neq 0 \tag{2.1}$$

$$[T_3^{(1)}, T_\pm^{(k)}] = [T_3^{(k)}, T_\pm^{(1)}] = \pm 2T_\pm^{(k)} \tag{2.2}$$

$$[T_+^{(1)}, T_-^{(k)}] = [T_+^{(k)}, T_-^{(1)}] = T_3^{(k)} \tag{2.3}$$

$$[T_0^{(n)}, T_0^{(m)}] = 0 \qquad \text{(for any } m, n) \tag{2.4}$$

$$[T_0^{(1)}, T_\alpha^{(n)}] = [T_0^{(n)}, T_\alpha^{(1)}] = 0 \qquad (\alpha \neq 0) \tag{2.5}$$

$$[T_0^{(2)}, T_\pm^{(2)}] = \pm(T_3^{(1)}T_\pm^{(2)} - T_3^{(2)}T_\pm^{(1)}) \tag{2.6}$$

$$[T_0^{(2)}, T_3^{(2)}] = 2(T_+^{(1)}T_-^{(2)} - T_+^{(2)}T_-^{(1)}) \tag{2.7}$$

$$[T_\alpha^{(m)}, T_0^{(n)}] = [T_\alpha^{(n)}, T_0^{(m)}] \tag{2.8}$$

$$[T_\pm^{(m)}, [T_3^{(2)}, T_\pm^{(n)}]] \pm [T_\pm^{(m)}, T_0^{(n)}]T_\pm^{(1)} = 0 \tag{2.9}$$

$$2[T_3^{(m)}, [T_+^{(n)}, T_-^{(2)}]] + [T_3^{(m)}, T_0^{(n)}]T_3^{(1)} = 0 \tag{2.10}$$

$$T_\pm^{(n+1)} = \tfrac{1}{2}\{\pm[T_3^{(2)}, T_\pm^{(n)}] + T_0^{(n)}T_\pm^{(1)} - T_0^{(1)}T_\pm^{(n)}\} \tag{2.11}$$

$$T_3^{(n+1)} = [T_+^{(n)}, T_-^{(n)}] + \tfrac{1}{2}(T_0^{(n)}T_3^{(1)} - T_0^{(1)}T_3^{(n)}). \tag{2.12}$$

In comparison with Drinfeld's theorem [11, 12] equations (2.9) and (2.10) are the consequence of equations (2.11) and (2.12). Actually, on the basis of the relation $[T_\alpha^{(n)}, T_\beta^{(m)}] = [T_\alpha^{(m)}, T_\beta^{(n)}](m < n, n \geqslant 3, \alpha, \beta = \pm, 3)$, one can show (see below) that the set satisfying equations (2.11) and (2.12) satisfy equations (2.9) and (2.10) order by order.

We emphasized that the constraint relations for $T_0^{(m)}$ come from the expansion of RTT relation that give rise to the new form of the coproduct $\triangle(T_{ab}) = \sum_c T_{ac} \otimes T_{cb}$. For the Y(SL(2)) case we have $T_0^{(2)} = (Q_0)^2$, however for Y(gl(2)) it is not the case. In comparison with the Drinfeld theory [10, 11] the inclusion of $T_0^{(m)}$ still preserves the property of mapping Y(gl(2)) into $T_{ab}^{(n)}$ provided $T_0^{(m)}$ satisfy equations (2.2)–(2.7).

When $m = n = 2$, taking equations (2.6) and (2.7) into account, equations (2.9) and (2.10) become

$$[T_\pm^{(2)}, [T_3^{(2)}, T_\pm^{(2)}]] = (T_3^{(1)}T_\pm^{(2)} - T_3^{(2)}T_\pm^{(1)})T_\pm^{(1)} \tag{2.13}$$

$$[T_3^{(2)}, [T_+^{(2)}, T_-^{(2)}]] = (T_+^{(1)}T_-^{(2)} - T_+^{(2)}T_-^{(1)})T_3^{(1)}. \tag{2.14}$$

From equations (2.1)–(2.3), (2.13), (2.14) we obtain

$$\begin{aligned} &2[T_\pm^{(2)}, [T_+^{(2)}, T_-^{(2)}]] \pm [T_3^{(2)}, [T_3^{(2)}, T_\pm^{(2)}]] \\ &\quad = 2(T_+^{(1)}T_-^{(2)} - T_+^{(2)}T_-^{(1)})T_\pm^{(1)} \pm (T_3^{(1)}T_\pm^{(2)} - T_3^{(2)}T_\pm^{(1)})T_3^{(1)}. \end{aligned} \tag{2.15}$$

With the relations for $T_\pm^{(1)}$ and $T_3^{(1)}$ (the Lie algebra) they form Y(gl(2)) algebra with the generators $T_\alpha^{(1)}$ and $T_\alpha^{(2)}$ ($\alpha = \pm, 3$) which can be realized through the operators Q_0^a and

Q_1^a in equations (1.10) and (1.11) for the Lie algebra gl(2). The determination of $T_0^{(2)}$, given by equations (2.6) and (2.7), is due to the consistence for equations (2.11), (2.12) and $[T_\alpha^{(n)}, T_\beta^{(m)}] = [T_\beta^{(n)}, T_\alpha^{(m)}] (m < n, n \geqslant 3)$.

Obviously, $T_\alpha^{(1)}$ can be realized through the spin-$\frac{1}{2}$ operators $S_i^\alpha (\alpha = \pm, 3)$ where $S_i^\pm = \sigma_i^\pm$ and $S_i^3 = \frac{1}{2}\sigma_i^3$ at the ith site, and $S_i^\pm = S_i^1 \pm i S_i^2$, σ_i are Pauli matrices. S_i^α $(\alpha = \pm, 3)$ satisfy the following relations:

$$[S_i^\alpha, S_j^\beta] = 0 \tag{2.16}$$

$$[S_i^3, S_j^\pm] = \pm\delta_{ij} S_i^\pm \tag{2.17}$$

$$(S_i^\pm)^2 = 0 \qquad (S_i^3)^2 = \tfrac{1}{4}. \tag{2.18}$$

By setting

$$Q_0^\alpha = \sum_i S_i^\alpha \qquad (\alpha = \pm, 3) \tag{2.19}$$

$$Q_1^\pm = \mp {\sum_{i,j}}' w_{ij} S_i^\pm S_j^3 \tag{2.20}$$

$$Q_1^3 = \frac{1}{2} {\sum_{i,j}}' w_{ij} S_i^+ S_j^- \tag{2.21}$$

where

$$T_\pm^{(l+1)} = Q_l^\pm = Q_l^1 \pm \mathrm{i} Q_l^{(2)} \qquad (l = 0, 1)$$

they are the same as those of equations (1.10) and (1.11), namely,

$$\begin{aligned} T_\pm^{(1)} &= Q_0^\pm = \sum_i S_i^\pm \\ T_3^{(1)} &= Q_0^3 = 2\sum_i S_i^3 \\ T_\pm^{(2)} &= Q_1^\pm = \mp {\sum_{i,j}}' w_{ij} S_i^\pm S_j^3 \\ T_3^{(2)} &= 2Q_1^3 = {\sum_{i,j}}' w_{ij} S_i^+ S_j^-. \end{aligned} \tag{2.22}$$

We can verify that $T_\alpha^{(1)}$ and $T_\alpha^{(2)}$ $(\alpha = \pm, 3)$ satisfy equations (2.1)–(2.3), (2.13)–(2.15), i.e. all the Yangian relations. The coproduct can be defined by $\triangle T_{ij} = \sum_k T_{ik} \otimes T_{kj}$.

When $n > 2$, the operators $T_\alpha^{(n)}$ $(\alpha = \pm, 3)$ can be obtained by using the recurrent formulae (2.11) and (2.12). In the absence of $T_0^{(2)}$ this is guaranteed by Drinfeld's theory [11, 12]. However, in the involusion of $T_0^{(n)}$ the situation becomes a little complicated. As we emphasized that in order to make the recurrent relations, equations (2.11) and (2.12), valid for $n \geqslant 3$, the $T_0^{(2)}$ and $T_0^{(n)} (n \geqslant 3)$ should be restricted by equations (2.6)–(2.10). The operator $T_0^{(2)}$ plays a basic role since it satisfies equations (2.6) and (2.7) and guarantees all the relations of RTT for $n \leqslant 3$. In terms of $T_0^{(2)}$ and taking equations (2.11), (2.12) and the set (2.8)–(2.10) into account the $T_0^{(n)} (n \geqslant 3)$ can then be sufficiently determined. By substituting $T_0^{(n)} (n \geqslant 2)$ into (1.25), C_n can be obtained. Hence, for the ready $T_0^{(2)}$ we expect to derive the Hamiltonian family for H–S model equations (1.1)–(1.3).

In one word, it is necessary to find suitable $T_0^{(n)}$ which satisfy all the requirements equations (2.6)–(2.10). When all of the $T_\alpha^{(n)} (\alpha = \pm, 3, 0)$ are ready we can use them to construct C_n that form a conserved family and commute with $T_0^{(n)}$. Of course, C_n commute with Yangian, but $T_0^{(n)}$ do not.

3. The construction of C_n

3.1. n = 2

We list all the independent equations for the RTT relation

$$[T_0^{(2)}, T_\pm^{(2)}] = \pm(T_3^{(1)}T_\pm^{(2)} - T_3^{(2)}T_\pm^{(1)}) \tag{3.1}$$

$$[T_0^{(2)}, T_3^{(2)}] = 2(T_+^{(1)}T_-^{(2)} - T_+^{(2)}T_-^{(1)}) \tag{3.2}$$

$$[T_3^{(2)}, [T_+^{(2)}, T_-^{(2)}]] = (T_+^{(1)}T_-^{(2)} - T_+^{(2)}T_-^{(1)})T_3^{(1)} \tag{3.3}$$

$$[T_\pm^{(2)}, [T_3^{(2)}, T_\pm^{(2)}]] = (T_3^{(1)}T_\pm^{(2)} - T_3^{(2)}T_\pm^{(1)})T_\pm^{(1)} \tag{3.4}$$

$$[T_+^{(1)}, T_-^{(2)}] = [T_+^{(2)}, T_-^{(1)}] = T_3^{(2)} \tag{3.5}$$

$$[T_3^{(1)}, T_\pm^{(2)}] = [T_3^{(2)}, T_\pm^{(1)}] = \pm 2T_\pm^{(2)}. \tag{3.6}$$

Noting that $T_0^{(1)} \equiv \operatorname{tr} T^{(1)}$ and

$$[T_0^{(1)}, T_{ab}^{(1)}] = 0. \tag{3.7}$$

The quantum determinant for $n = 2$ reads:

$$C_2 = T_0^{(2)} - \tfrac{1}{2}\{\tfrac{1}{2}T_3^{(1)}T_3^{(1)} + T_+^{(1)}T_-^{(1)} + T_-^{(1)}T_+^{(1)}\} + \tfrac{1}{2}[T_0^{(1)} + (\tfrac{1}{2}T_0^{(1)})^2] \tag{3.8}$$

where $T_0^{(2)}$ satisfies equations (3.1) and (3.2).

To solve equations (3.1) and (3.2), we suppose

$$T_0^{(2)} = {\sum_{i,j}}' f_{ij}(P_{ij} - 1) \tag{3.9}$$

where

$$P_{ij} = 2S_i^3 S_j^3 + S_i^+ S_j^- + S_i^- S_j^+ + \tfrac{1}{2} \tag{3.10}$$

(it is easy to check that $P_{ij}S_i^\alpha S_j^\beta = S_j^\alpha S_i^\beta P_{ij}$) and f_{ij} are constants to be determined.

By substituting equation (3.9) into equations (3.1) and (3.2), we find the sufficient solution:

$$f_{ij} = \tfrac{1}{2}(w_{ij})^2 \tag{3.11}$$

so that

$$T_0^{(2)} = \tfrac{1}{2}{\sum_{i,j}}'(w_{ij})^2(P_{ij} - 1). \tag{3.12}$$

Putting $T_0^{(2)}$ into equation (3.8) and setting

$$C_2 = -2H_2 + A_2 \tag{3.13}$$

where

$$A_2 = \tfrac{1}{2}(T_0^{(1)} + \tfrac{1}{2}(T_0^{(1)})^2) + \sum_{i,j}\tfrac{1}{4} + \sum_i \tfrac{1}{2} \tag{3.14}$$

is formed by constants and the lower-order Casimir, one obtains

$$H_2 = {\sum_{i,j}}' \frac{z_i z_j}{z_{ij} z_{ji}}(P_{ij} - 1)$$

that is just (1.1).

3.2. $n = 3$

We list all the relations

$$T_\pm^{(3)} = \tfrac{1}{2}\{\pm[T_3^{(2)}, T_\pm^{(2)}] + T_0^{(2)}T_\pm^{(1)} - T_0^{(1)}T_\pm^{(2)}\} \tag{3.15}$$

$$T_3^{(3)} = [T_+^{(2)}, T_-^{(2)}] + \frac{1}{2}(T_0^{(2)}T_3^{(1)} - T_0^{(1)}T_3^{(2)}) \tag{3.16}$$

$$[T_0^{(3)}, T_0^{(2)}] = 0 \tag{3.17}$$

$$[T_0^{(2)}, T_\alpha^{(3)}] = [T_0^{(3)}, T_\alpha^{(2)}] \qquad (\alpha = \pm, 3) \tag{3.18}$$

$$[T_\alpha^{(2)}, T_\alpha^{(3)}] = 0 \tag{3.19}$$

$$[T_\alpha^{(2)}, T_\beta^{(3)}] = [T_\alpha^{(3)}, T_\beta^{(2)}] \qquad (\alpha, \beta \neq 0) \tag{3.20}$$

$$[T_0^{(2)}, T_\pm^{(3)}] = \pm(T_3^{(1)}T_\pm^{(3)} - T_3^{(3)}T_\pm^{(1)}) \tag{3.21}$$

$$[T_-^{(1)}, T_+^{(3)}] = -T_3^{(3)} \qquad [T_3^{(1)}, T_+^{(3)}] = 2T_+^{(3)} \tag{3.22}$$

$$[T_-^{(1)}, T_3^{(3)}] = 2T_-^{(3)}. \tag{3.23}$$

The C_3 is given by

$$C_3 = T_0^{(3)} + \tfrac{1}{2}T_0^{(1)}T_0^{(2)} - \{\tfrac{1}{2}T_3^{(1)}T_3^{(2)} + T_+^{(1)}T_-^{(2)} + T_-^{(1)}T_+^{(2)}\} + C_2. \tag{3.24}$$

We first prove that suppose all the relations for $n = 2$ are satisfied then $T_\alpha^{(3)}$ $(\alpha = \pm, 3)$ determined by equations (3.15) and (3.16) satisfy equations (3.19)–(3.23). As an example we check $[T_3^{(3)}, T_+^{(2)}] = [T_3^{(2)}, T_+^{(3)}]$. From equation (3.15) it follows

$$\begin{aligned}[T_3^{(3)}, T_+^{(2)}] - [T_3^{(2)}, T_+^{(3)}] &= [[T_+^{(2)}, T_-^{(2)}], T_+^{(2)}] - \tfrac{1}{2}[T_3^{(2)}, [T_3^{(2)}, T_+^{(2)}]] \\ &\quad + \tfrac{1}{2}[T_0^{(2)}, T_+^{(2)}]T_3^{(1)} - \tfrac{1}{2}[T_3^{(2)}, T_0^{(2)}]T_+^{(1)}.\end{aligned} \tag{3.25}$$

Taking $T_+^{(2)} = \frac{1}{2}[T_3^{(2)}, T_+^{(1)}]$ into account the first term on the RHS of equation (3.25) becomes

$$[[T_+^{(2)}, T_-^{(2)}], T_+^{(2)}] = \tfrac{1}{2}[[T_+^{(2)}, T_-^{(2)}], (T_3^{(2)}T_+^{(1)} - T_+^{(1)}T_3^{(2)})]$$

which by virtue of equations (2.4)–(2.7) and equations (3.3) and (3.4) is equal to

$$\tfrac{1}{2}[T_3^{(2)}, T_0^{(2)}]T_+^{(1)} + \tfrac{1}{2}[T_3^{(2)}, [T_3^{(2)}, T_+^{(2)}]] - \tfrac{1}{4}[[T_0^{(2)}, T_3^{(2)}], T_+^{(1)}]T_3^{(1)}.$$

Using Jacobi identity, $[T_+^{(1)}, T_0^{(2)}] = 0$ and $T_+^{(2)} = \frac{1}{2}[T_3^{(2)}, T_+^{(1)}]$ one obtains

$$[[T_0^{(2)}, T_3^{(2)}], T_+^{(1)}] = 2[T_0^{(2)}, T_+^{(2)}].$$

Hence one has

$$[[T_+^{(2)}, T_-^{(2)}], T_+^{(2)}] = \tfrac{1}{2}[T_3^{(2)}, T_0^{(2)}]T_+^{(1)} + \tfrac{1}{2}[T_3^{(2)}, [T_3^{(2)}, T_+^{(2)}]] - \tfrac{1}{2}[T_0^{(2)}, T_+^{(2)}]T_3^{(1)}$$

that leads to (3.25) $= 0$. Similarly one can prove that equations (3.19)–(3.23) hold. This conclusion is nothing but the direct check of Drinfelds theorem in the case $n = 3$. Because there is not the explicit statement in [11, 12] in the presence of $T_0^{(n)}(T_0^{(n)}(n \geqslant 2)$ are not Casimir), the direct check here does make sense itself.

Now for given $T_0^{(2)}$ we should find $T_0^{(3)}$ such that it satisfies equations (3.17) and (3.18) provided $T_\alpha^{(3)}$ $(\alpha = \pm, 3)$ are given by equations (3.15) and (3.16). We shall prove that the form of

$$T_0^{(3)} = -\tfrac{1}{18}\sum_{i,j,k}{}' w_{ij}w_{jk}w_{ki}(P_{ijk} - 1) - \tfrac{1}{2}T_0^{(1)}T_0^{(2)} \qquad (T_0^{(1)} = C_1) \tag{3.26}$$

satisfies equations (3.17) and (3.18).

If equation (3.26) holds, by substituting it into equation (3.24) one obtains

$$C_3 = -\tfrac{4}{9}H_3 + A_3 \tag{3.27}$$

where $A_3 = C_2$. The H_3 is exactly the Hamiltonian equation (1.2).

Now let us verify that equation (3.17) is satisfied, for example, for $[T_0^{(3)}, T_0^{(2)}] = 0$. Equations (3.12) can be recasted to

$$T_0^{(2)} = \tfrac{1}{2}\sum_{i,j}{}'(P_{ij} - 1) - 2H_2$$

and

$$H_3 = \tfrac{1}{8}\sum_{i,j,k}{}'(w_{ij}w_{jk}w_{ki} + w_{ij} + w_{jk} + w_{ki})(P_{ijk} - 1)$$

respectively. In terms of equations (3.26) and (3.27), $T_0^{(3)}$ can be rewritten in the form:

$$T_0^{(3)} = \tfrac{1}{6}\sum_{i,j,k}{}'w_{ij}(P_{ijk} - 1) - \tfrac{4}{9}H_3 - \tfrac{1}{2}T_0^{(1)}T_0^{(2)}. \tag{3.28}$$

In view of $T_0^{(1)} = C_1$, we obtain the simplified form of the LHS of equation (3.17):

$$\begin{aligned}
[T_0^{(2)}, T_0^{(3)}] = {} & \tfrac{1}{12}\left[\sum_{l,m}{}'P_{lm}, \sum_{i,j,k}{}'w_{ij}P_{ijk}\right] - \tfrac{2}{9}\left[\sum_{l,m}{}'P_{lm}, H_3\right] \\
& + \tfrac{1}{4}C_1\left[\sum_{l,m}{}'P_{lm}, T_0^{(2)}\right] - \tfrac{1}{3}\left[H_2, \sum_{i,j,k}{}'w_{ij}P_{ijk}\right] \\
& + C_1[H_2, T_0^{(2)}] + \tfrac{8}{9}[H_2, H_3].
\end{aligned} \tag{3.29}$$

Let us compute the RHS of the above expression.

Using

$$\tfrac{1}{2}\sum_{i,j}{}'(P_{ij} - 1) = \tfrac{1}{4}(T_3^{(1)})^2 + \tfrac{1}{2}(T_+^{(1)}T_-^{(1)} + T_-^{(1)}T_+^{(1)}) - \tfrac{1}{2}\sum_{i,j}{}'1$$

$$\tfrac{1}{6}\sum_{i,j,k}{}'w_{ij}(P_{ijk} - 1) = \tfrac{1}{2}T_3^{(1)}T_3^{(2)} + T_+^{(1)}T_-^{(2)} + T_-^{(1)}T_+^{(2)}$$

and equation (1.8) and the relationship between $T_0^{(2)}$ and H_2, it is not difficult to prove that the last five terms on the RHS of equation (3.29) vanish.

Next we only need to prove that the first term on the RHS of equation (3.29) is equal to zero. On account of

$$\begin{aligned}
P_{ij}P_{jk} = {} & \tfrac{1}{2}(P_{ij} + P_{jk} + P_{ki} - 1) + S_i^3(S_j^+S_k^- - S_j^-S_k^+) + S_j^3(S_k^+S_i^- - S_k^-S_i^+) \\
& + S_k^3(S_i^+S_j^- - S_i^-S_j^+)
\end{aligned}$$

we obtain

$$\sum_{i,j,k}{}'w_{ij}P_{ijk} = 3\sum_{i,j,k}{}'w_{ij}P_{ij}P_{jk} = 6\sum_{i,j,k}{}'(w_{ij} + w_{jk} + w_{ki})S_i^3S_j^+S_k^-.$$

Hence

$$\tfrac{1}{3}\left[\sum_{l,m}{}'P_{lm}, \sum_{i,j,k}{}'w_{ij}P_{ijk}\right]$$

$$= 4\left\{ \sum_{i,j,k,m}{}' [(P_{im} + P_{jm} + P_{km}), (w_{ij} + w_{jk} + w_{ki}) S_i^3 S_j^+ S_k^-] \right.$$
$$\left. + \sum_{i,j,k}{}' [(P_{ij} + P_{jk} + P_{ki}), (w_{ij} + w_{jk} + w_{ki}) S_i^3 S_j^+ S_k^-] \right\}.$$

The direct calculation shows that the first commutation vanishes because

$$\sum_{i,j,k,m}{}' (w_{ij} + w_{jk} + w_{ki})$$
$$\times(-S_i^+ S_m^- S_j^+ S_k^- + S_i^- S_j^+ S_k^- S_m^+ + 2S_i^3 S_j^+ S_k^3 S_m^- - 2S_i^3 S_j^3 S_k^- S_m^+) = 0$$

where equations (2.16)–(2.18) have been used. Similarly, the second commutator also vanishes, we thus verify that $[T_0^{(3)}, T_0^{(2)}] = 0$.

Next let us prove equation (3.18). As an example, let us consider $\alpha = 3$, i.e. one needs to prove

$$[T_0^{(3)}, T_3^{(2)}] = [T_0^{(2)}, T_3^{(3)}]. \tag{3.30}$$

On account of

$$T_3^{(3)} = [T_+^{(2)}, T_-^{(2)}] + \tfrac{1}{2}(T_0^{(2)} T_3^{(1)} - T_0^{(1)} T_3^{(2)}) \tag{3.31}$$

and substituting equations (3.26) and (3.31) into equation (3.30), then equation (3.30) becomes

$$-\tfrac{1}{18}\left[\sum_{i,j,k}{}' w_{ij} w_{jk} w_{ki} (P_{ijk} - 1), T_3^{(2)}\right] = [T_0^{(2)}, [T_+^{(2)}, T_-^{(2)}]]. \tag{3.32}$$

Substituting equations (3.1), (3.5) and (3.6) into the RHS of equation (3.32) successively it becomes:

$$\text{RHS of (3.32)} = [T_3^{(1)}, [T_+^{(2)}, T_-^{(2)}]] + 2(T_+^{(2)} T_-^{(2)} - T_-^{(2)} T_+^{(2)}) + [T_-^{(2)}, T_3^{(2)}] T_+^{(1)}$$
$$+ [T_+^{(2)}, T_3^{(2)}] T_-^{(1)}. \tag{3.33}$$

Noting that

$$[T_+^{(2)}, T_-^{(2)}] = \sum_{i,j,k}{}' w_{ij} w_{jk} S_i^3 (2S_j^3 S_k^3 + S_j^+ S_k^- + S_j^- S_k^+) - \tfrac{1}{2} \sum_{i,j}{}' (w_{ij})^2 S_j^3 \tag{3.34}$$

which leads to

$$[[T_+^{(2)}, T_-^{(2)}], T_3^{(1)}] = 0 \tag{3.35}$$

hence, the first term of equation (3.33) vanishes. Further, using

$$[(\tfrac{1}{2} T_3^{(2)} T_3^{(1)} + T_+^{(2)} T_-^{(1)} + T_-^{(2)} T_+^{(1)}), T_3^{(2)}]$$
$$= [T_+^{(2)}, T_3^{(2)}] T_-^{(1)} + [T_-^{(2)}, T_3^{(2)}] T_+^{(1)} + 2(T_+^{(2)} T_-^{(2)} - T_-^{(2)} T_+^{(2)})$$

we obtain

$$[T_0^{(2)}, [T_+^{(2)}, T_-^{(2)}]] = [(\tfrac{1}{2} T_3^{(2)} T_3^{(1)} + T_+^{(2)} T_-^{(1)} + T_-^{(2)} T_+^{(1)}), T_3^{(2)}]. \tag{3.36}$$

Making a comparison between equation (3.32) and equation (3.36) and taking the expression for $\sum'_{i,j,k} w_{ij}(P_i jk - 1)$ into account we conclude that equation (3.18) is identified with the requirement:

$$[(\tfrac{1}{18} \sum_{i,j,k}{}' w_{ij} w_{jk} w_{ki} + \tfrac{1}{6} \sum_{i,j,k} w_{ij}) P_{ijk}, T_3^{(2)}] = 0 \tag{3.37}$$

that is nothing but $[H_3, T_3^{(2)}] = 0$. It was verified in [3, 5] for $T_3^{(2)} \sim Q_1^{(3)}$. Of course, (3.37) can also be directly checked. In the same manner, one finds that equation (3.18) holds for $\alpha = \pm, 3$.

The other relations for equations (3.19) and (3.20) can be verified in a similar manner.

3.3. n = 4

The RTT relations are given by

$$[T_0^{(4)}, T_0^{(2)}] = 0 \tag{3.38}$$

$$[T_0^{(4)}, T_\alpha^{(2)}] = [T_0^{(2)}, T_\alpha^{(4)}] \qquad (\alpha = \pm, 3) \tag{3.39}$$

$$[T_0^{(4)}, T_0^{(3)}] = 0 \tag{3.40}$$

$$[T_0^{(4)}, T_\alpha^{(m)}] = [T_0^{(m)}, T_\alpha^{(4)}] \qquad (\alpha = \pm, 3 m = 2, 3) \tag{3.41}$$

$$[T_0^{(4)}, T_\alpha^{(1)}] = 0 \tag{3.42}$$

together with equations (2.11) and (2.12) with $n = 3$. Using equations (2.2)–(2.10) for $n, m \leqslant 3$, we have

$$[T_0^{(2)}, T_\pm^{(m)}] = \pm(T_3^{(1)} T_\pm^{(m)} - T_3^{(m)} T_\pm^{(1)}) \tag{3.43}$$

$$[T_0^{(2)}, T_3^{(m)}] = 2(T_+^{(1)} T_-^{(m)} - T_+^{(m)} T_-^{(1)}) \qquad (m = 3, 4) \tag{3.44}$$

$$[T_+^{(1)}, T_-^{(m)}] = [T_+^{(m)}, T_-^{(1)}] = T_3^{(m)} \tag{3.45}$$

$$[T_3^{(1)}, T_\pm^{(m)}] = \pm 2 T_\pm^{(m)} \qquad [T_3^{(m)}, T_\pm^{(1)}] = \pm 2 T_\pm^{(m)} \tag{3.46}$$

$$[T_-^{(2)}, T_+^{(4)}] = [T_-^{(4)}, T_+^{(2)}]. \tag{3.47}$$

The relations shown by equations (3.43)–(3.47) are not concerned with $T_0^{(4)}$. They can be checked by making use of equations (2.11) and (2.12) and the knowledge of sections 3.1 and 3.2. Therefore the key relations concerning $T_0^{(4)}$ that have to be checked are equations (3.38)–(3.42).

Equation (1.25) takes the form:

$$C_4 = T_0^{(4)} + \tfrac{1}{2} T_0^{(2)} + \tfrac{1}{2}(3C_3 - C_2) + \tfrac{1}{2}(T_0^{(1)} T_0^{(3)} - T_3^{(1)} T_3^{(3)}) - (T_+^{(1)} T_-^{(3)} + T_-^{(1)} T_+^{(3)}) \\ + \tfrac{1}{4}[((T_0^{(2)})^2 - (T_3^{(2)})^2) - 2(T_+^{(2)} T_-^{(2)} + T_-^{(2)} T_+^{(2)})]. \tag{3.48}$$

We should find a suitable $T_0^{(4)}$ satisfying equations (3.38)–(3.42). The direct check is very difficult. Since the point of this paper is to give a new interpretation of the H–S model from the point of view of RTT relation, we can admit the derived results in [3–5] that will simplify the following calculation a lot. Using the results for $n = 2$ and $n = 3$ after a complicated calculation, equation (3.48) becomes

$$C_4 = T_0^{(4)} - \tfrac{1}{2} \sum_{i,j,k,m}{}' w_{ij} w_{jk} P_{im} P_{jk} - \tfrac{1}{8} \sum_{i,j,k,m}{}' w_{ij} w_{km} P_{im} P_{jk} - \tfrac{3}{4} (T_0^{(2)})^2 \\ - \tfrac{7}{8} \sum_{i,j,k}{}' w_{ij} w_{jk} P_{ik} + \tfrac{1}{2} \sum_{i,j,k}{}' (w_{ij})^2 P_{ik} + A T_0^{(2)} + B + T_0^{(1)} T_0^{(3)} \tag{3.49}$$

where $A = \frac{N}{2} - \frac{1}{2} T_0^{(1)} + C_2 - \frac{3}{4}$ and $B = \frac{11}{8} \sum'_{i,j} (w_{ij})^2 + \frac{1}{2}(3C_3 - C_2) - \frac{1}{2} C_1 (C_3 - C_2)$.

We propose that $T_0^{(4)}$ takes the form:

$$T_0^{(4)} = -\tfrac{1}{32} \sum_{i,j,k,m}{}' (w_{ij} w_{jk} w_{km} w_{mi} + 1)(P_{ijkm} - 1)$$

$$+\tfrac{1}{4}\sum_{i,j,k,m}{}'(w_{ij}w_{jk}P_{ik}P_{jm}-\tfrac{1}{2}w_{ij}w_{km}P_{im}P_{jk})$$
$$-\sum_{i,j,k}{}'[\tfrac{1}{8}(2N-11)w_{ij}w_{jk}+\tfrac{3}{4}(w_{ij})^2]P_{ik}-A'T_0^{(2)}-T_0^{(1)}T_0^{(3)}$$
$$-\tfrac{1}{2}H_4''-\tfrac{3}{4}(T_0^{(2)})^2-\tfrac{1}{2}H_4''' \tag{3.50}$$

where

$$A'=A-\frac{1}{2}=\frac{N}{2}-\frac{1}{2}T_0^{(1)}+C_2-\frac{5}{4}$$
$$H_4''=-2\sum_{i,j}{}'\left(\frac{z_iz_j}{z_{ij}z_{ji}}\right)^2(P_{ij}-1)$$
$$H_4'''=-\tfrac{1}{3}H_2.$$

If equation (3.50) is true, then we have:

$$C_4=\tfrac{1}{2}H_4+A_4 \tag{3.51}$$

where $A_4=2\sum'_{i,j}(w_{ij})^2-\frac{3}{8}N\sum'_{i,j}(w_{ij})^2+\frac{1}{2}(3C_3-C_2)-\frac{1}{2}C_1(C_3-C_2)$ which is conserved, and H_4 takes the same form as that given in equation (1.3). Equation (3.13) means that

$$C_2=-2H_2+A_2$$
$$=T_0^{(2)}-\tfrac{1}{2}\{\tfrac{1}{2}T_3^{(1)}T_3^{(1)}+T_+^{(1)}T_-^{(1)}+T_-^{(1)}T_+^{(1)}\}+\tfrac{1}{2}\{T_0^{(1)}+\tfrac{1}{2}(T_0^{(1)})^2\}. \tag{3.52}$$

Therefore it holds

$$[H_4,T_0^{(2)}]=0 \tag{3.53}$$

and combining equation (3.49) it leads to:

$$[C_4,T_0^{(2)}]=0. \tag{3.54}$$

Similarly, after lengthy calculations we can check for the $T_0^{(4)}$:

$$[C_n,T_0^{(m)}]=0 \qquad (m,n=2,3,4) \tag{3.55}$$

hence

$$[C_n,T_\alpha^{(m)}]=0 \qquad (m,n=2,3,4). \tag{3.56}$$

On the basis of knowledge of [3–5] in equations (1.8) and (1.9) we shall give the verification of equations (3.38)–(3.42) for the proposed $T_0^{(4)}$ at the same level of the validity discussed in [3]. Because the calculations are lengthy we only give a sketch of the computation.

3.3.1. The verification of equation (3.38). Substituting $[C_4,T_0^{(2)}]$ into equation (3.48) and using equations (3.21), (3.1) and (3.2) we have:

$$[T_0^{(4)},T_0^{(2)}]-\{T_3^{(1)}(T_+^{(3)}T_-^{(1)}-T_+^{(1)}T_-^{(3)})+T_+^{(1)}(T_3^{(1)}T_-^{(3)}-T_3^{(3)}T_-^{(1)})$$
$$+T_-^{(1)}(T_3^{(3)}T_+^{(1)}-T_3^{(1)}T_+^{(3)})+\tfrac{1}{2}T_3^{(2)}(T_+^{(2)}T_-^{(1)}-T_+^{(1)}T_-^{(2)})$$
$$+\tfrac{1}{2}(T_+^{(2)}T_-^{(1)}-T_+^{(2)}T_-^{(2)})T_3^{(2)}+\tfrac{1}{2}T_+^{(2)}(T_3^{(1)}T_-^{(2)}-T_3^{(2)}T_-^{(1)})$$
$$+\tfrac{1}{2}(T_3^{(1)}T_-^{(2)}-T_3^{(2)}T_-^{(1)})T_+^{(2)}+\tfrac{1}{2}T_-^{(2)}(T_3^{(2)}T_+^{(1)}-T_3^{(1)}T_+^{(2)})$$
$$+\tfrac{1}{2}(T_3^{(2)}T_+^{(1)}-T_3^{(1)}T_+^{(2)})T_-^{(2)}\}=0.$$

By a straightforward calculation on account of equations (2.1)–(2.12) for $n = 2$ we find that the parenthesis in the above relation is equal to zero, thus one obtains:

$$[T_0^{(4)}, T_0^{(2)}] = 0.$$

3.3.2. The verification of equation (3.39). From equations (3.54) and (3.46) it follows

$$\begin{aligned}
&[T_0^{(4)}, T_+^{(2)}] + \tfrac{1}{2}[T_0^{(2)}, T_+^{(2)}] + \tfrac{1}{2}T_0^{(1)}[T_0^{(3)}, T_+^{(2)}] - \tfrac{1}{2}[T_3^{(1)}, T_+^{(2)}]T_3^{(3)} - \tfrac{1}{2}T_3^{(1)}[T_3^{(2)}, T_+^{(2)}] \\
&\quad -T_+^{(1)}[T_-^{(3)}, T_+^{(2)}] - [T_-^{(1)}, T_+^{(2)}]T_+^{(3)} + \tfrac{1}{4}T_0^{(2)}[T_0^{(2)}, T_+^{(2)}] + \tfrac{1}{4}[T_0^{(2)}, T_+^{(2)}]T_0^{(2)} \\
&\quad -\tfrac{1}{4}T_3^{(2)}[T_3^{(2)}, T_+^{(2)}] - \tfrac{1}{4}[T_3^{(2)}, T_+^{(2)}]T_3^{(2)} - \tfrac{1}{2}T_+^{(2)}[T_-^{(2)}, T_+^{(2)}] \\
&\quad -\tfrac{1}{2}[T_-^{(2)}, T_+^{(2)}]T_+^{(2)} = 0.
\end{aligned} \tag{3.57}$$

Using equations (2.1)–(2.12) for $m \leqslant 3$ and $n = 2$, after straightforward calculation equation (3.57) becomes:

$$\begin{aligned}
&[T_0^{(4)}, T_+^{(2)}] + \tfrac{1}{2}T_0^{(1)}[T_0^{(2)}, T_+^{(3)}] - \tfrac{1}{2}T_3^{(1)}[T_3^{(2)}, T_+^{(3)}] - \tfrac{1}{2}T_3^{(1)}[T_3^{(2)}, T_+^{(3)}] \\
&\quad -T_+^{(1)}[T_-^{(2)}, T_+^{(3)}] + [T_3^{(2)}, T_+^{(3)}] = 0.
\end{aligned} \tag{3.58}$$

On the other hand, it holds

$$\begin{aligned}
[T_0^{(2)}, T_+^{(4)}] &= [T_0^{(2)}, \tfrac{1}{2}[T_3^{(2)}, T_+^{(3)}]] - \tfrac{1}{2}T_0^{(1)}[T_0^{(2)}, T_+^{(3)}] \\
&= T_+^{(1)}[T_-^{(2)}, T_+^{(3)}] + T_+^{(2)}[T_+^{(3)}, T_-^{(1)}] - \tfrac{1}{2}T_0^{(1)}[T_0^{(2)}, T_+^{(3)}] \\
&\quad +\tfrac{1}{2}[T_3^{(2)}, [T_0^{(2)}, T_+^{(3)}]].
\end{aligned} \tag{3.59}$$

Using equations (3.42) and (3.44), equation (3.59) becomes:

$$[T_0^{(2)}, T_+^{(4)}] = T_+^{(1)}[T_-^{(2)}, T_+^{(3)}] + [T_+^{(2)}, T_3^{(3)}] - \tfrac{1}{2}T_0^{(1)}[T_0^{(2)}, T_+^{(3)}] + \tfrac{1}{2}T_3^{(1)}[T_3^{(2)}, T_+^{(3)}]. \tag{3.60}$$

Comparing equation (3.60) with equation (3.58) we obtain

$$[T_0^{(2)}, T_+^{(4)}] = [T_0^{(4)}, T_+^{(2)}]. \tag{3.61}$$

In a similar manner we can prove:

$$[T_0^{(2)}, T_-^{(4)}] = [T_0^{(4)}, T_-^{(2)}] \tag{3.62}$$

$$[T_0^{(2)}, T_3^{(4)}] = [T_0^{(4)}, T_3^{(2)}]. \tag{3.63}$$

3.3.3. The verification of equation (3.40). From equations (3.24) and (3.53) it follows

$$[T_0^{(4)}, T_0^{(3)}] = \tfrac{1}{2}T_3^{(1)}[T_0^{(4)}, T_3^{(2)}] + T_+^{(1)}[T_0^{(4)}, T_-^{(2)}] + T_-^{(1)}[T_0^{(4)}, T_+^{(2)}]. \tag{3.64}$$

Using equations (3.38), (3.62), (3.63) and (3.44) for $m = 4$, equation (3.64) becomes:

$$\begin{aligned}
[T_0^{(4)}, T_0^{(3)}] &= 2T_+^{(1)}T_-^{(4)} + 2T_-^{(1)}T_+^{(4)} + [T_+^{(1)}, T_-^{(1)}]T_3^{(4)} - 2T_+^{(1)}T_-^{(4)} \\
&\quad -2T_-^{(1)}T_+^{(4)} + T_3^{(1)}[T_-^{(1)}, T_+^{(4)}] \\
&= T_3^{(1)}T_3^{(4)} - T_3^{(1)}T_3^{(4)} = 0.
\end{aligned} \tag{3.65}$$

3.3.4. The verification of equation (3.41) for $m = 3$. By virtue of equations (3.24) and (3.54) we obtain:

$$
\begin{aligned}
[T_0^{(3)}, T_+^{(4)}] &= \tfrac{1}{2}[T_3^{(1)}, T_+^{(4)}]T_3^{(2)} + \tfrac{1}{2}T_3^{(1)}[T_3^{(2)}, T_+^{(4)}] + T_+^{(1)}[T_-^{(2)}, T_+^{(4)}] + [T_-^{(1)}, T_+^{(4)}]T_+^{(2)} \\
&\quad - \tfrac{1}{2}T_0^{(1)}[T_0^{(2)}, T_+^{(4)}] \\
&= T_+^{(4)}T_3^{(2)} + \tfrac{1}{2}T_3^{(1)}[T_3^{(2)}, T_+^{(4)}] + T_+^{(1)}[T_-^{(2)}, T_+^{(4)}] - T_3^{(4)}T_+^{(2)} \\
&\quad - \tfrac{1}{2}T_0^{(1)}[T_0^{(2)}, T_+^{(4)}]
\end{aligned}
\qquad (3.66)
$$

where $[T_+^{(2)}, T_+^{(4)}] = 0$ has been used (this relation can be checked).

On the other hand,

$$
\begin{aligned}
[T_0^{(4)}, T_+^{(3)}] &= [T_0^{(4)}, \tfrac{1}{2}([T_3^{(2)}, T_+^{(2)}] + T_0^{(2)}T_+^{(1)} - T_0^{(1)}T_+^{(2)})] \\
&= T_+^{(1)}[T_-^{(4)}, T_+^{(2)}] + T_+^{(4)}T_3^{(2)} + \tfrac{1}{2}T_3^{(1)}[T_3^{(2)}, T_+^{(4)}] - T_3^{(4)}T_+^{(2)} \\
&\quad - \tfrac{1}{2}T_0^{(1)}[T_0^{(2)}, T_+^{(4)}]
\end{aligned}
\qquad (3.67)
$$

where $[T_0^{(4)}, T_+^{(2)}] = [T_0^{(2)}, T_+^{(4)}]$ has been used, which can be directly checked so that

$$[T_0^{(3)}, T_+^{(4)}] - [T_0^{(4)}, T_+^{(3)}] = T_+^{(1)}[T_-^{(2)}, T_+^{(4)}] - T_+^{(1)}[T_-^{(4)}, T_+^{(2)}]. \qquad (3.68)$$

Because of equations (3.38) and (3.47) one has:

$$[T_0^{(3)}, T_+^{(4)}] = [T_0^{(4)}, T_+^{(3)}]. \qquad (3.69)$$

In a similar way, we can prove:

$$[T_0^{(3)}, T_-^{(4)}] = [T_0^{(4)}, T_-^{(3)}] \qquad (3.70)$$

$$[T_0^{(3)}, T_3^{(4)}] = [T_0^{(4)}, T_3^{(3)}]. \qquad (3.71)$$

Finally, Equation (3.42) can be easily proved by using equation (3.48). Therefore it has been verified that the proposed $T_0^{(4)}$ satisfies all the requirements by the relations equations (3.38)–(3.42). Such a $T_0^{(4)}$ gives rise to

$$C_4 = \tfrac{1}{2}H_4 + A_4 \qquad (3.72)$$

where A_4 commutes with the Yangian and so H_4 does at the level in [3].

4. Conclusion

As was pointed out in [3, 5], the Hamiltonian set of the H–S model (1.1)–(1.4) possesses Yangian symmetry. In the above discussion we have shown that the spin-$\frac{1}{2}$ Hamiltonian family can be derived from the RTT relation through the operators $T_0^{(n)} (2 \leqslant n \leqslant 4)$, namely, the $H_n (n \leqslant 4)$ are related to the quantum determinant of the transfer matrix C_n. The set C_n forms a conserved family, so does $T_0^{(n)}$. They commute with each other. $T_0^{(n)}$ do not commute with the Yangian, but they are related to the Bethe ansatz approach following the quantum inverse scattering methods in the diagonalization.

We start from equations (1.19) and (1.16) to find the C_n ($n \leqslant 4$) which are not constants, because the $T_0^{(n)}$ cannot be expressed by the Q_0^{α} and $Q_1^{\alpha} (\alpha = \pm, 3)$ themselves. Rather, they are expressed in terms of the 'more fundamental' operators S_i^{α} which form the Yangian. By virtue of equation (1.25) C_n are determined by $T_0^{(n)}$ and the Yangian operators $T_\alpha^{(1)}, T_\alpha^{(2)} (\alpha = \pm, 3)$. The $T_0^{(2)}$ is constrained by the consistence between equations (2.11) and (2.12) for $n = 3$ and equations (2.1)–(2.7). With such $T_0^{(2)}$ the higher ordered $T_0^{(n)}$ $(2 \leqslant n \leqslant 4)$ can be determined by satisfying equations (2.4)–(2.8). This process is quite like that made in discussing the long-ranged interaction model in [9]. With the obtained $T_0^{(n)}$ the set C_n can be found. The Hamiltonian family of the H–S model is thus formed by C_n. Of course, the solutions for $T_0^{(n)}$ are sufficient. This approach may be viewed as the generalization of the idea presented in [6]. It is general and dependent on the particular realization of $T_0^{(2)}$. It is expected that new candidates will be found, then new models may be constructed.

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